Last time

diagonal matrices (square)

$$\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & a_n \end{pmatrix}$$

(add and multiply component-wise)

triangular matrices (square) $\longrightarrow$ upper

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

$\longrightarrow$ lower

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{pmatrix}$$

transpose matrices of any (even rectangular) matrix

$$A \in \mathbb{R}^{m\times n} \xrightarrow{\text{swap rows and columns}} A^T \in \mathbb{R}^{n\times m}$$

$$\begin{pmatrix} 2 & 3 \\ 5 & 6 \\ 9 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 5 & 9 \\ 3 & 6 & -1 \end{pmatrix}$$

Properties : $(A+B)^T = A^T + B^T$, $\underline{(AB)^T = B^T A^T}$

New topic: inverses: if A exists,

$$AB = AC \xrightarrow[\text{left by } A^{-1}]{\text{multiply on}} \underbrace{A^{-1}A}_{=Id}B = \underbrace{A^{-1}A}_{Id}C \Longrightarrow B = C$$

$$BA = CA \xrightarrow[\text{right by } A^{-1}]{\text{multiply on}} B\overbrace{AA^{-1}}^{Id} = C\overbrace{AA^{-1}}^{Id} \Longrightarrow B = C$$

DEF 9.1: $\underset{\mathbb{R}^{m\times n}}{\overset{}{A}}$ is invertible if $\exists$ $\underset{\mathbb{R}^{n\times m}}{A^{-1}}$ such that

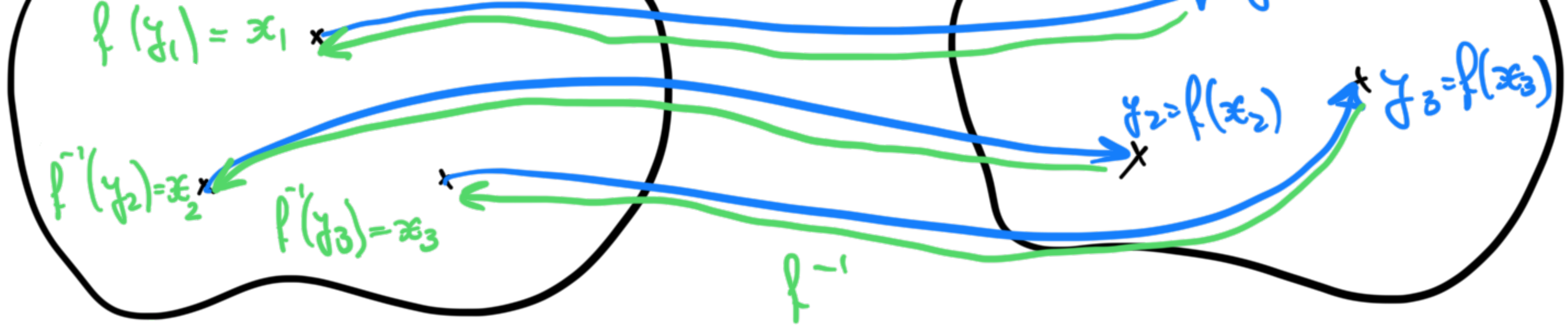
$$AA^{-1} = I_m \quad \text{and} \quad A^{-1}A = I_n$$

Linear functions:

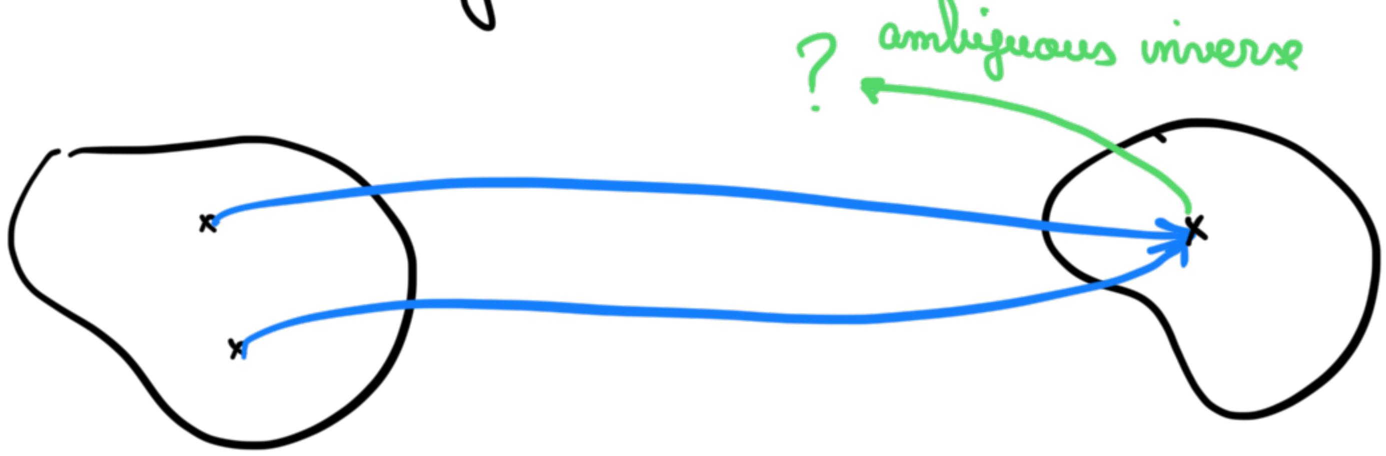
$$\underset{m\times n}{A} \rightsquigarrow f: \mathbb{R}^n \to \mathbb{R}^m$$
$$f(x) = Ax$$

$$\underset{n\times m}{A^{-1}} \rightsquigarrow f^{-1}: \mathbb{R}^m \to \mathbb{R}^n$$
$$f^{-1}(x) = A^{-1}x$$

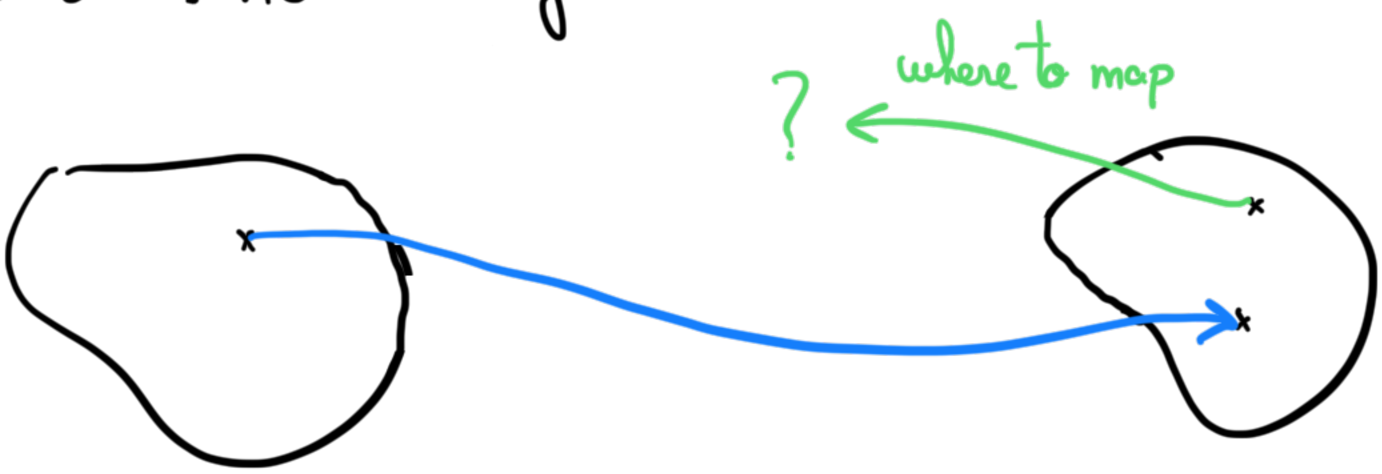
$$f \circ f^{-1} = Id_{\mathbb{R}^m} \quad \text{and} \quad f^{-1} \circ f = Id_{\mathbb{R}^n}$$



f which is not injective cannot be invertible, e.g.



f which is not surjective cannot be invertible



THM 9.2: A function is bijective $\Longleftrightarrow$ it is invertible

In particular, a linear function $f: \mathbb{R}^n \to \mathbb{R}^m$ has a chance of being bijective/invertible only if $m=n$; however, this may not be enough

Example: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f =$ rotation by angle α $\rightsquigarrow$ $f^{-1} =$ rotation by angle $-\alpha$

Example: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$f =$ dilation by λ $\rightsquigarrow$ $f^{-1} =$ dilation by $\frac{1}{\lambda}$

Example: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$f =$ projection onto x-axis $\rightsquigarrow$ f^{-1} does not exist (not bijective)

Examples above via matrices

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \neq I_2, \quad \forall a, b, c, d$$

DEF 9.3: We call a $n \times n$ matrix A **invertible**

if $\exists\, A^{-1} \in \mathbb{R}^{n \times n}$ s.t. $AA^{-1} = I_n = A^{-1}A$

any of these implies the other

<u>Prop</u>: if A has an inverse, then it is unique.

suppose it has two inverses : A^{-1} and $\tilde{A}^{-1}$

$$AA^{-1} = A^{-1}A = I_n \qquad AA^{-1} = I_n \quad \Big| \text{ mult on left by } \tilde{A}^{-1}$$

$$A\tilde{A}^{-1} = \tilde{A}^{-1}A = I$$

$$AA^{-1} = A^{-1}A = I_n$$

$$\underbrace{\tilde{A}^{-1}A}_{=I_n}A^{-1} = \tilde{A}^{-1} \Rightarrow A^{-1} = \tilde{A}^{-1}$$

Prop: $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A^{-1}, B^{-1} exist)

this follows from the equalities

$$ABB^{-1}A^{-1} = AA^{-1} = I_n$$

$$B^{-1}A^{-1}AB = B^{-1}B = I_n$$

Caveat: $(A+B)^{-1}$ might not exist even if A^{-1}, B^{-1} exist and it does not have any reasonable formula in terms of A^{-1}, B^{-1}

Prop: $(\lambda A)^{-1} = \lambda^{-1}A^{-1}$ for any $\lambda \in \mathbb{R}\setminus 0$

Prop: $(A^T)^{-1} = (A^{-1})^T$

this follows from the equalities

$$A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$$

$$(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n$$

Prop: $(A^{-1})^{-1} = A$

because $AA^{-1} = A^{-1}A = I_n$

Prop: $(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}$

just apply first proposition $k-1$ times

Upshot: any product of invertible matrices (and their transposes) is also invertible

Ex: $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{a_n} \end{pmatrix}$ exists iff $a_1, \ldots, a_n \neq 0$

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_1} & & *' \\ & \ddots & \\ 0 & & \frac{1}{a_n} \end{pmatrix}$$

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ * & & a_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a_1} & & 0 \\ & \ddots & \\ *' & & \frac{1}{a_n} \end{pmatrix}$$

e.g. $\begin{pmatrix} 2 & 5 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & ? \\ 0 & \frac{1}{3} \end{pmatrix}$ must have $2 \cdot ? + \frac{5}{3} = 0$

New topic: Inverses are intimately related with solving equations

New topic: Inverses and invertibility

$$AX = B$$

$$\Big\Downarrow \text{ multiply on left by } A^{-1}$$

$$X = \underbrace{A^{-1}A}_{I}X = A^{-1}B$$

Let's explore the connection between inverses and Gaussian elimination

① $A = \begin{pmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ & \vdots & \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \vdots & \\ \lambda a_{i1} & \cdots & \lambda a_{in} \\ & \vdots & \end{pmatrix} = D_i^{(\lambda)} A$

② $A = \begin{pmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ & \vdots & \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \vdots & \\ a_{j1} & \cdots & a_{jn} \\ a_{i1} & \cdots & a_{in} \\ & \vdots & \end{pmatrix} = S_{ij} A$

③ $A = \begin{pmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ a_{j1} & \cdots & a_{jn} \\ & \vdots & \end{pmatrix} \rightsquigarrow \begin{pmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ a_{j1}+\lambda a_{i1} & \cdots & a_{jn}+\lambda a_{in} \\ & \vdots & \end{pmatrix} = T_{ji}^{(\lambda)} A$

New topic: all three of the row operations above can be realized by multiplying A on left by a suitable matrix

$$\text{①}\quad i\begin{pmatrix} \ddots & & & & 0 \\ & 1 & & & \\ & & \lambda & & \\ & & & 1 & \\ 0 & & & & \ddots \; 1 \end{pmatrix} = D_i^{(\lambda)}$$

$$\text{②}\quad \begin{pmatrix} \ddots & & & & & & 0 \\ & 1 & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & 1 \ddots 1 & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & 1 & \\ 0 & & & & & & \ddots \; 1 \end{pmatrix} = S_{ij}$$

$$\text{③}\quad \begin{array}{c c} & \\ \text{row } j & \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & & \ddots & & \\ \cdots \lambda & & 0 & \ddots & \\ & & & & 1 \end{pmatrix} \\ & \text{column } i \end{array} = T_{ji}^{(\lambda)}$$

these are called elementary matrices

Let's work out above in an example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}}_{D_1^{(\lambda)} =} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_{12} =} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow \underbrace{\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}}_{T_{21}^{(\lambda)} =} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21}+\lambda a_{11} & a_{22}+\lambda a_{12} \end{pmatrix}$$

Inverses of elementary matrices are also elementary matrices

$$\left(D_i^{(\lambda)}\right)^{-1} = D_i^{\left(\frac{1}{\lambda}\right)} \qquad \left(S_{ij}\right)^{-1} = S_{ij} \qquad \left(T_{ji}^{(\lambda)}\right)^{-1} = T_{ji}^{(-\lambda)}$$

Gaussian elimination for a square matrix A. Either

- $REF(A)$ has $< n$ pivots $\Rightarrow$ no chance of being invertible (corresponding function is not bijective)
- $REF(A)$ has n pivots $\Rightarrow$ $REF(A) = I_n$; in this case

$$A \rightsquigarrow M_1 A \rightsquigarrow M_2 M_1 A \rightsquigarrow \cdots \rightsquigarrow M_k \cdots M_2 M_1 A = I_n$$

for various $M_1, \ldots, M_k \in \{ D_i^{(\lambda)}, S_{ij}, T_{ji}^{(\lambda)} \}$

$$\Longrightarrow A^{-1} = M_k \cdots M_1$$ (effective way to compute inverse as a product of elementary matrices)

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ let's calculate A^{-1} by Gaussian elimination

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ \boxed{1} & 3 & 1 \end{pmatrix} \xrightarrow{\text{swap rows 1 and 3}} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{subtract } 2\times \text{row 1 from row 2}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \rightsquigarrow S_{13} A \rightsquigarrow T_{21}^{(-2)} S_{13} A$$

$$\xrightarrow{\text{multiply row 2 by } -1} \begin{pmatrix} \boxed{1} & 3 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{subtract } 3\times \text{row 2 from row 1}} \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

$$\rightsquigarrow D_2^{(-1)} T_{21}^{(-2)} S_{13} A \rightsquigarrow T_{12}^{(-3)} D_2^{(-1)} T_{21}^{(-2)} S_{13} A$$

$$\xrightarrow[\text{from row 2}]{\text{subtract row 3}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{to row 1}]{\text{add } 2\times\text{row 3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$\rightsquigarrow T_{23}^{(-1)} T_{12}^{(-3)} D_2^{(-1)} T_{21}^{(-2)} S_{13} A \rightsquigarrow T_{13}^{(2)} T_{23}^{(-1)} T_{12}^{(-3)} D_2^{(-1)} T_{21}^{(-2)} S_{13} A$$

upshot:

$$T_{13}^{(2)} T_{23}^{(-1)} T_{12}^{(-3)} D_2^{(-1)} T_{21}^{(-2)} S_{13} A = I_3$$

$$\Downarrow$$

$$A^{-1} = T_{13}^{(2)} T_{23}^{(-1)} T_{12}^{(-3)} D_2^{(-1)} T_{21}^{(-2)} S_{13}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & -5 \\ -1 & -1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

elementary matrices

In general, if you know $A = M_1 \cdots M_k$

then the solution to $AX = B$

$$\Updownarrow$$

$$M_1 \cdots M_k X = B$$

is $X = \underbrace{M_k^{-1} \cdots M_1^{-1}}_{\text{also elementary matrices}} B$

Let's spell out the 3×3 elementary matrices

$$D_1^{(\lambda)} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D_2^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D_3^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$S_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S_{13} = \begin{pmatrix} 0 & & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_{21}^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{31}^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix} \qquad T_{32}^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix}$$

$$T_{12}^{(\lambda)} = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{13}^{(\lambda)} = \begin{pmatrix} 1 & 0 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{23}^{(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$$